HL Paper 3

Let c be a positive, real constant. Let G be the set $\{x \in \mathbb{R} | -c < x < c\}$. The binary operation $*$ is defined on the set G by $x * y = \frac{x+y}{1+\frac{xy}{c^2}}$.

a. Simplify $\frac{c}{2} * \frac{3c}{4}$. [2]

b. State the identity element for G under $*$. [1]

c. For $x \in G$ find an expression for x^{-1} (the inverse of x under $*$). [1]

d. Show that the binary operation $*$ is commutative on G . [2]

e. Show that the binary operation $*$ is associative on G . [4]

f. (i) If $x,\ y \in G$ explain why $(c-x)(c-y) > 0$. [2]

(ii) Hence show that $x + y < c + \frac{xy}{c}$.

g. Show that G is closed under $*$. [2]

h. Explain why $\{G, *\}$ is an Abelian group. [2]

Markscheme

a. $\frac{c}{2} * \frac{3c}{4} = \frac{\frac{c}{2}+\frac{3c}{4}}{1+\frac{1}{2}\cdot\frac{3}{4}}$ ***M1***

$= \frac{\frac{5c}{4}}{\frac{11}{8}} = \frac{10c}{11}$ ***A1***

[2 marks]

b. identity is 0 ***A1***

[1 mark]

c. inverse is –*x* ***A1***

[1 mark]

d.

$x * y = \frac{x+y}{1+\frac{xy}{c^2}},\ y * x = \frac{y+x}{1+\frac{yx}{c^2}}$ ***M1***

(since ordinary addition and multiplication are commutative)

$x * y = y * x$ so $*$ is commutative ***R1***

Note: Accept arguments using symmetry.

[2 marks]

e.

$(x * y) * z = \frac{x+y}{1+\frac{xy}{c^2}} * z = \frac{\left(\frac{x+y}{1+\frac{xy}{c^2}}\right)+z}{1+\left(\frac{x+y}{1+\frac{xy}{c^2}}\right)\frac{z}{c^2}}$ ***M1***

$$= \frac{\frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\frac{xy}{c^2}\right)}}{\frac{\left(1+\frac{xy}{c^2}+\frac{xz}{c^2}+\frac{yz}{c^2}\right)}{\left(1+\frac{xy}{c^2}\right)}} = \frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\left(\frac{xy+xz+yz}{c^2}\right)\right)} \quad \textbf{\textit{A1}}$$

$$x * (y * z) = x * \left(\frac{y+z}{1+\frac{yz}{c^2}}\right) = \frac{x+\left(\frac{y+z}{1+\frac{yz}{c^2}}\right)}{1+\frac{x}{c^2}\left(\frac{y+z}{1+\frac{yz}{c^2}}\right)}$$

$$= \frac{\frac{\left(x+\frac{xyz}{c^2}+y+z\right)}{\left(1+\frac{yz}{c^2}\right)}}{\frac{\left(1+\frac{yz}{c^2}+\frac{xy}{c^2}+\frac{xz}{c^2}\right)}{\left(1+\frac{yz}{c^2}\right)}} = \frac{\left(x+y+z+\frac{xyz}{c^2}\right)}{\left(1+\left(\frac{xy+xz+yz}{c^2}\right)\right)} \quad \textbf{\textit{A1}}$$

since both expressions are the same $*$ is associative ***R1***

Note: After the initial ***M1A1***, correct arguments using symmetry also gain full marks.

[4 marks]

f. (i) $c > x$ and $c > y \Rightarrow c - x > 0$ and $c - y > 0 \Rightarrow (c-x)(c-y) > 0$ ***R1AG***

(ii) $c^2 - cx - cy + xy > 0 \Rightarrow c^2 + xy > cx + cy \Rightarrow c + \frac{xy}{c} > x + y$ (as $c > 0$)

so $x + y < c + \frac{xy}{c}$ ***M1AG***

[2 marks]

g. if $x,\ y \in G$ then $-c - \frac{xy}{c} < x + y < c + \frac{xy}{c}$

thus $-c\left(1 + \frac{xy}{c^2}\right) < x + y < c\left(1 + \frac{xy}{c^2}\right)$ and $-c < \frac{x+y}{1+\frac{xy}{c^2}} < c$ ***M1***

(as $1 + \frac{xy}{c^2} > 0$) so $-c < x * y < c$ ***A1***

proving that G is closed under $*$ ***AG***

[2 marks]

h. as $\{G, *\}$ is closed, is associative, has an identity and all elements have an inverse ***R1***

it is a group ***AG***

as $*$ is commutative ***R1***

it is an Abelian group ***AG***

[2 marks]

Examiners report

a. Most candidates were able to answer part (a) indicating preparation in such questions. Many students failed to identify the command term “state” in parts (b) and (c) and spent a lot of time – usually unsuccessfully - with algebraic methods. Most students were able to offer satisfactory solutions to part (d) and although most showed that they knew what to do in part (e), few were able to complete the proof of associativity. Surprisingly few managed to answer parts (f) and (g) although many who continued to this stage, were able to pick up at least one of the marks for part (h), regardless of what they had done before. Many candidates interpreted the question as asking to prove that the group was Abelian, rather than proving that it was an Abelian group. Few were able to fully appreciate the significance in part (i) although there were a number of reasonable solutions.

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A random variable X has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{1}{4} & 1 \le x < 3 \\ 0 & x \ge 3 \end{cases}$$

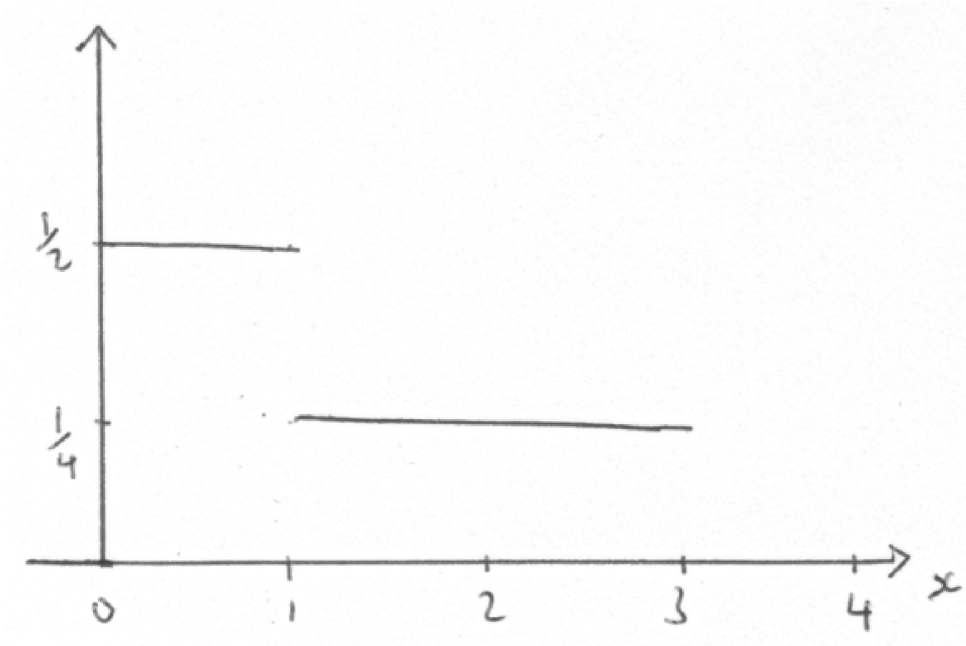
a. Sketch the graph of $y = f(x)$. [1]

b. Find the cumulative distribution function for X. [5]

c. Find the interquartile range for X. [3]

Markscheme

a.



A1

Note: Ignore open / closed endpoints and vertical lines.

Note: Award ***A1*** for a correct graph with scales on both axes and a clear indication of the relevant values.

[1 mark]

b.
$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < 1 \\ \frac{x}{4} + \frac{1}{4} & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

considering the areas in their sketch or using integration ***(M1)***

$F(x) = 0,\ x < 0,\ F(x) = 1,\ x \ge 3$ ***A1***

$F(x) = \frac{x}{2},\ 0 \le x < 1$ ***A1***

$F(x) = \frac{x}{4} + \frac{1}{4},\ 1 \le x < 3$ ***A1A1***

Note: Accept $<$ for $\le$ in all places and also $>$ for $\ge$ first ***A1***.

[5 marks]

c. $Q_3 = 2,\ Q_1 = 0.5$ ***A1A1***

$\text{IQR is } 2 - 0.5 = 1.5$ ***A1***

[3 marks]

Total [9 marks]

Examiners report

a. Part (a) was correctly answered by most candidates. Some graphs were difficult to mark because candidates drew their lines on top of the ruled lines in the answer book. Candidates should be advised not to do this. Candidates should also be aware that the command term ‘sketch’ requires relevant values to be indicated.

b. In (b), most candidates realised that the cumulative distribution function had to be found by integration but the limits were sometimes incorrect.

c. In (c), candidates who found the upper and lower quartiles correctly sometimes gave the interquartile range as $[0.5,\ 2]$. It is important for candidates to realise that that the word range has a different meaning in statistics compared with other branches of mathematics.